Heat-kernel expansion on non compact domains and a generalised zeta-function regularisation procedure

Guido Cognola [*,a], Emilio Elizalde [†,b,c] and Sergio Zerbini [‡,a]

[a]Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare,
Gruppo Collegato di Trento, Italia

[b]Consejo Superior de Investigaciones Científicas
Instituto de Ciencias del Espacio (ICE/CSIC)
Campus UAB, Facultat de Ciències, Torre C5-Parell-2a planta,
08193 Bellaterra (Barcelona), Spain

[c] Institut d'Estudis Espacials de Catalunya (IEEC),
Edifici Nexus, Gran Capità 2-4, 08034 Barcelona, Spain

Abstract: *Heat-kernel expansion and zeta function regularisation are discussed for Laplace type operators with discrete spectrum in non compact domains. Since a general theory is lacking, the heat-kernel expansion is investigated by means of several examples. It is pointed out that for a class of exponential (analytic) interactions, generically the non-compactness of the domain gives rise to logarithmic terms in the heat-kernel expansion. Then, a meromorphic continuation of the associated zeta function is investigated. A simple model is considered, for which the analytic continuation of the zeta function is not regular at the origin, displaying a pole of higher order. For a physically meaningful evaluation of the related functional determinant, a generalised zeta function regularisation procedure is proposed.*

PACS numbers: *02.30.Tb, 02.70.Hm, 04.62.+v*

Keywords: Heat-kernel expansion, Zeta function-regularisation, functional determinants, confining potentials.

1 Introduction

Within the so-called one-loop approximation in quantum field theory, the Euclidean one-loop effective action can be expressed in terms of the sum of the classical action and a contribution depending on a functional determinant of an elliptic differential operator, the so called fluctuation operator. The ultraviolet one-loop divergences which are present need to be regularised by means of a suitable technique (for recent reviews, see [1]–[5]).

*e-mail: *cognola@science.unitn.it*
†e-mail: *elizalde@ieec.uab.es*
‡e-mail: *zerbini@science.unitn.it*

In general, one works in Euclidean spacetime and deals with a self-adjoint, non-negative, second-order differential operator of the form

$$L = -\Delta \, + V \,, \tag{1.1}$$

where Δ is the Laplace-Beltrami operator and V a potential depending on the classical background solution and containing, in general, a mass term. It is well known that the one-loop effective action $W \equiv W[\Phi]$, is related to the functional determinant of the field operator L by

$$W = -\ln Z = S_c + \frac{1}{2} \ln \det \frac{L}{\mu^2} \,, \tag{1.2}$$

S_c being the classical action and μ^2 a renormalisation parameter, which appears for dimensional reasons.

The one-loop divergences may be dealt with by using a variant of the zeta-function regularisation method [6]-[8]. One namely introduces the regularisation parameter ε and considers

$$W(\varepsilon) = S - \frac{1}{2} \int_0^\infty dt \, \frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)} \, \mathrm{Tr} \, e^{-tL/\mu^2} = S - \frac{1}{2\varepsilon} \zeta(\varepsilon|L/\mu^2) \,, \tag{1.3}$$

where, as usual, for the elliptic operator L the zeta function is defined by means of the Mellin-like transform

$$\zeta(s|L) = \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} \, \mathrm{Tr} \, e^{-tL} \,, \qquad\qquad \zeta(s|L/\mu^2) = \mu^{2s} \zeta(s|L) \,. \tag{1.4}$$

Here the heat trace $\mathrm{Tr}\, e^{-tL}$ plays a preeminent role. Recall that, for a second-order elliptic non-negative operator L in a *compact* $d-$dimensional manifold without boundary, one has the small-t asymptotic expansion

$$\mathrm{Tr} \, e^{-tL} \simeq \sum_{j=0}^\infty A_j(L) \, t^{j-d/2} \,, \tag{1.5}$$

where $A_j(L)$ are the Seeley-DeWitt coefficients [9, 10]. As a result, for a second-order differential operator in $d-$dimensions, the integral (1.4) is convergent in the domain $\mathrm{Re}\, s > \frac{d}{2}$.

In the compact case, $\zeta(s|L)$ is regular at the origin and one has the well known result $\zeta(0|L) = A_{d/2}(L)$. The latter quantity is computable (see for example the recent reviews [11, 12]) and depends on the potential and on geometric invariants. In particular, for odd dimensional manifolds without boundaries, $\zeta(0|L) = 0$. Performing a Taylor expansion of the zeta function we obtain

$$W(\varepsilon) = S - \frac{1}{2\varepsilon} \zeta(0|L) + \frac{\zeta(0|L)}{2} \ln \mu^2 + \frac{\zeta'(0|L)}{2} + O(\varepsilon) \,. \tag{1.6}$$

Thus, the one-loop divergences as well as finite contributions to the one-loop effective action are expressed in terms of the zeta function and its derivative evaluated at the origin.

In this paper, we would like to discuss a more general situation where *logarithmic terms* in the heat-trace asymptotics are present. First, we recall a well known but crucial fact concerning the local heat-kernel expansion associated with a Laplace type operator in R^d of the kind

$$H = -\Delta \, + V(x) \,. \tag{1.7}$$

If the potential is real and non-negative, with an additional, rather mild hypothesis, the operator H is essentially self-adjoint in $C_0^\infty(\mathrm{R}^d)$. We will be interested in confining potentials which, with the additional hypothesis of being also smooth functions, give rise to a discrete spectrum. As has been shown in Refs. [13, 14], the local heat-kernel expansion can be partially summed over and rewritten under the form

$$K_t(x,x) = \frac{1}{(4\pi t)^{d/2}} e^{-tV(x)} \sum_{n=0}^{\infty} b_n(x)\, t^n\,, \tag{1.8}$$

where the new coefficients $b_n(x)$ can easily be computed and depend only on the derivatives of the potential $V(x)$. The first few read

$$b_0(x) = 1\,, \qquad b_1(x) = 0\,,$$

$$b_2(x) = -\frac{1}{6}\Delta V\,, \qquad b_3(x) = -\frac{\Delta^2 V}{60} + \frac{\nabla_k V \nabla_k V}{12}\,, \tag{1.9}$$

$$b_4(x) = -\frac{\Delta^3 V}{840} + \frac{(\Delta V)^2}{72} + \frac{\nabla_i\nabla_j V \nabla_i \nabla_j V}{90} + \frac{\nabla_k V \nabla_k \Delta V}{30}\,. \tag{1.10}$$

We will make use of such a re-summation for obtaining the heat-kernel trace asymptotics in the next section.

If one is dealing with smooth compact manifolds, the passage to the heat-kernel trace is accomplished by integrating term by term over the coordinates, and no logarithmic contribution in the heat-trace expansion appears. However, in the case of non-smooth manifolds one may get logarithmic terms in the heat-kernel trace, for example when one considers the Laplace operator on higher-dimensional cones [15, 16], but also in 4-dimensional spacetimes with a 3-dimensional, non-compact, hyperbolic spatial section of finite volume [17], and in the case of general pseudo-differential operators [18]. More recently, the presence of logarithmic terms in self-interacting scalar field theory defined on manifolds with non-commutative coordinates has also been pointed out [19]-[21]. This goes together with a non-typical behaviour of the corresponding zeta function: possibly a simple pole at the origin and higher-order poles at other places.

Here we would like to investigate the case of Laplace-type self-adjoint operators defined on *non-compact* manifolds. To our knowledge, for the general case of a confining potential and discrete spectrum, a systematic theory has yet to be formulated. A pioneering investigation along this line can be found in [22]. Other studies involve one dimensional problems on the real half-line [23] and the Barnes zeta functions [24]. With regard to the presence of logarithmic terms in heat-trace asymptotics, one should note that they have been considered in the abstract context of regularised products in many places.

Recall that, under certain conditions, the regularised product associated with an infinite sequence of non-zero complex numbers $\{\lambda_n\}$ has a related Dirichlet series $\sum_n \lambda_n^{-s}$ (the zeta-function). In this paper, we are only interested in the case when the λ_n are eigenvalues of *non-negative* differential operators and the zeta function converges absolutely for $\mathrm{Re}\, s$ sufficiently large. When this zeta-function is holomorphic at the origin, the regularised product is defined as $\exp\left[-\zeta'(0)\right]$. A general theory is presented in [27, 28] and other relevant papers are [29]-[31] and references quoted therein. It is worth mentioning, that the case of non-compact domains but with scattering potentials, namely the ones for which a continuous spectrum exists, is well understood and the $S-$matrix or the phase shift function then enter the game (see, for instance, [32]). In this context, delta-like potentials have also been considered (see for example [33]-[35], and references therein). If the potential is singular, for instance, proportional to $1/x^2$, the

presence of logarithmic terms in the local heat-kernel expansion is also possible, their coefficients becoming distributions [25] (see also the recent paper [26] and references therein). Here, we will not deal with situations of this kind. Moreover, for the sake of simplicity, we will limit ourselves to the R^d flat case.

The content of the paper is as follows. In Sect. 2, local heat-kernel asymptotics are reviewed for the simple case we are going to deal with. In Sect. 3, the heat-trace asymptotics are investigated and the possibility for the presence of logarithmic terms is pointed out explicitly. The consequences of such unusual terms are discussed in Sect. 4 in some detail. Finally, in Sect. 5, a simple model of confinement is proposed and a generalisation of the zeta-function regularisation method is constructed to deal with this case. The paper ends with some conclusions and an Appendix.

2 Heat-kernel trace asymptotics in non-compact domains

In this section we will show that, for suitable classes of potentials in non-compact domains, logarithmic terms can actually be present in the heat-trace expansion. Under the usual hypothesis concerning the potential $V(x)$ —namely $V(x)$ smooth enough, non-negative and going to infinity for $|x| \to \infty$— the heat trace can be shown to exist and the heat kernel asymptotics is given by

$$\mathrm{Tr}\, e^{-tH} = \int_{R^d} dx\, K_t(x,x) = \frac{1}{(4\pi t)^{d/2}} \int_{R^d} dx\ e^{-tV(x)} \left[1 + t^2 b_2(x) + t^3 b_3(x) + \cdots\right] \tag{2.1}$$

In particular we shall focus our attention on the class of spherical potentials, $V(x) = V(r)$, $r \in [0,\infty)$, and thus

$$\mathrm{Tr}\, e^{-tH} = \frac{\Omega_d}{(4\pi t)^{d/2}} \int_0^\infty dr\ \ r^{d-1} e^{-tV(r)} \left[1 + O(t^2)\right] , \tag{2.2}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$. The latter expression will be our starting point for further discussions.

As a first family of potentials, let us consider $V(r)$ to be a positive polynomial of degree Q. In this case, we will show that logarithmic terms are absent, but the leading term goes as $O(t^{-d/2-d/Q})$, in contrast to the leading behaviour $O(t^{-d/2})$ associated with the compact case. To prove this, since we are interested in the short-t leading term, it is sufficient to consider the leading term of the potential, namely $V(r) = r^Q + \cdots$. Thus, the leading term in the heat trace reads

$$\mathrm{Tr}\, e^{-tH} \simeq \frac{\Omega_d}{(4\pi t)^{d/2}} \int_0^\infty dr\, r^{d-1} e^{-t\, r^Q} = \frac{2^{1-d}\Gamma(d/Q)}{Q\,\Gamma(d/2)\, t^{d/2+d/Q}} . \tag{2.3}$$

One can check that this result holds true for the case $Q = 2$, for which the heat trace is well known, since it corresponds to the partition function of a harmonic oscillator in $d-$dimensions. In fact one has eigenvalues $2n+1$ for each dimension and then

$$\mathrm{Tr}\, e^{-tH} = \left(\sum_0^\infty e^{-t(2n+1)}\right)^d = \frac{1}{(2\sinh t)^d} \simeq \frac{1}{(2t)^d} + \cdots , \tag{2.4}$$

in agreement with Eq. (2.3) (note that here $m = 1/2$ and $\omega = 2$). Our results also agree with those for the 1-dimensional case investigated in [23].

The situation drastically changes if one considers exponential confining potentials which, for large r, go asymptotically as $V(r) \simeq e^{r^Q}$. We will show that in these cases logarithmic terms are present. In fact, we have

$$\begin{aligned}
\mathrm{Tr}\, e^{-tH} &\simeq \frac{\Omega_d}{(4\pi)^{d/2} t^{d/2}} \int_0^\infty dr\, r^{d-1} e^{-te^{r^Q}} \\
&= \frac{\Omega_d}{(4\pi)^{d/2}\, t^{d/2} Q} \int_1^\infty dy\; y^{-1} e^{-ty} \left(\ln y\right)^{d/Q-1} .
\end{aligned} \tag{2.5}$$

For simplicity, let us now assume $\frac{d}{Q}$ to be an integer. In such case we can use (A.4) and (A.6) of the Appendix, thus obtaining the leading term in the form

$$\mathrm{Tr}\, e^{-tH} \simeq \frac{(-1)^{d/Q}}{2^d Q \Gamma(d/2+1)\, t^{d/2}} \left(\ln t\right)^{d/Q} . \tag{2.6}$$

It has to be noted that with respect to the compact case, for such class of potentials on non-compact manifolds, the leading term in the trace is modified by the presence of the logarithmic factor $(\ln t)^{d/Q}$. We also note that, for $Q = 1$, (2.6) yields the same result obtained by Nash in [22] using a different method. Let us emphasise that those comparisons are essential both for consistency reasons and in view of its application to real situations in physics.

Equations (2.3) and (2.5) give only the leading term in the trace of the heat kernel, but in principle it is possible to go on in the expansion by integrating other terms of the local asymptotics. However it should be stressed that more terms in the local expansion can give contributions of the same order to the trace asymptotics. This can be easily seen by considering, for instance, the 1-dimensional harmonic oscillator described by the Hamiltonian

$$H = -\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{4} , \qquad \hbar = 1 , \qquad m = \frac{1}{2} . \tag{2.7}$$

For this model, one has

$$\mathrm{Tr}\, e^{-tH} = \frac{1}{2\sinh(\omega t/2)} = \frac{1}{\omega t} - \frac{\omega t}{24} + O(t^3) \tag{2.8}$$

$$\begin{aligned}
K_t(x,x) &= \sqrt{\frac{\omega}{4\pi \sinh \omega t}}\, e^{-\omega x^2 (\cosh \omega t - 1)/2 \sinh \omega t} \\
&= \frac{e^{-tV}}{\sqrt{4\pi t}} \left(1 + b_2 t^2 + b_3 t^3 + \cdots \right) , \\
& \qquad b_2 = -\frac{\omega^2}{12} , \qquad b_3 = \frac{\omega^4 x^2}{48} .
\end{aligned} \tag{2.9}$$

In order to get the expansion (2.8) up to order t, one needs to integrate the local expansion up to order $t^{5/2}$. This means that both b_2 and b_3 give a contribution of order t in the trace asymptotics. To obtain the subsequent term t^3, one has to consider all b_n coefficients up to b_6.

3 Meromorphic extension of the zeta-function

With respect to the compact case, the meromorphic structure of the zeta function associated with the operator H is generically quite complicated and it is strictly related to the form of the

potential. In order to show this, we first consider the polynomial case $V(r) = r^Q$ and assume $Q = 2P$ to be an even number. Under such assumption all b_n coefficients are polynomials in r and the heat-trace asymptotics are of the form

$$\mathrm{Tr}\, e^{-tH} = \sum_n C_n\, t^{\alpha_n - (d/2+d/Q)} \,, \qquad C_0 = \frac{\Gamma(d/Q)}{2^{d-1} Q \Gamma(d/2)} \,, \qquad \alpha_0 = 1 < \alpha_1 < \alpha_2 < \cdots \quad (3.1)$$

where the C_n are numerical coefficients obtained by integrating the local expansion, and the α_n are rational numbers. Making use of (1.4) and splitting the integration over t into $(0, 1]$ and $[1, \infty)$, we get

$$\zeta(s|H) = \frac{1}{\Gamma(s)} \sum_n \frac{C_n}{s + \alpha_n - (\frac{d}{2} + \frac{d}{Q})} + \frac{J(s)}{\Gamma(s)} \,, \quad (3.2)$$

where $J(s)$ is an analytic function. It follows that for such class of potentials the zeta function admits only simple poles —as it happens in the compact case— but whose location strictly depends on the form of the potential, since the α_n are not universal powers. Moreover, we see that $\zeta(0|H)$ is not vanishing if and only if $\alpha_n = d/2 + d/Q$ for some n and the corresponding coefficient C_n is different from zero (note that if this coefficient $C_n = 0$, then the corresponding term is absent from the sum, for any s).

The situation becomes more complicated for the class of exponential potentials we have considered in the previous section. In fact, in such case one obtains in general an asymptotic expansion with terms of the kind $t^\alpha (\ln t)^\beta$, α and β being rational numbers which depend on the potential, and this means that the meromorphic extension of the zeta function will have poles or branch points of order β at $s = -\alpha$ (see (A.8) in the Appendix).

In order to compute the non-holomorphic structure of the zeta function for this class of potentials it is convenient to proceed as follows. We use the general expression (1.8) in (1.4) and thus, for $\mathrm{Re}\, s$ sufficiently large and $V(x) = V(r) > 0$, we can write

$$\begin{aligned} \zeta(s|H) &\sim \frac{1}{(4\pi)^{d/2}\Gamma(s)} \sum_n \int_0^\infty dt\, t^{s+n-d/2-1} \int_{R^d} dx\, b_n(x) e^{-tV(x)} \\ &= \sum_n \frac{\Gamma(s+n-d/2)}{(4\pi)^{d/2}\Gamma(s)} \int_{R^d} dx\, b_n(x) [V(x)]^{-(s+n-d/2)} \,, \end{aligned} \quad (3.3)$$

which is well defined for even Q, since in such case all coefficients $b_n(x)$ are regular everywhere. Since $V(r)$ is exponential like and spherically symmetric, we may assume that

$$b_n = \sum_{pq} C^n_{pq} r^p V^q \,, \qquad 0 \le p \le 2(n-1)(Q-1) \,, \qquad 1 \le q < n \,, \qquad n \ge 2 \,. \quad (3.4)$$

Now, the integration can be performed and we obtain for the non-holomorphic part

$$\begin{aligned} \zeta(s|H) &\sim \frac{\Omega_d}{(4\pi)^{d/2}\Gamma(s)} \Bigg[\Gamma(s-d/2) \int_0^\infty dr\, r^{d-1} e^{-(s-d/2)r^Q} \\ &\qquad + \sum_{n\ge 2;pq} C^n_{pq} \Gamma(s+n-d/2) \int_0^\infty dr\, r^{d+p-1} e^{-(s+n-q-d/2)r^Q} \Bigg] \\ &= \frac{\Omega_d}{(4\pi)^{d/2} Q \Gamma(s)} \Bigg[\frac{\Gamma(s-d/2)\Gamma(d/Q)}{(s-d/2)^{d/Q}} \end{aligned}$$

$$+\sum_{n\geq 2;pq} C^n_{pq}\frac{\Gamma(s+n-d/2)\Gamma((d+p)/Q)}{(s+n-q-d/2)^{(d+p)/Q}}\Bigg] \,. \tag{3.5}$$

As a consequence, it follows that generically the zeta function may have poles and branch points of any order. It should also be noted that in the case of even dimension it is not holomorphic at the origin.

For example, in the simplest case $d = Q = 2$, $V(r) = e^{r^2}$, by straightforward dimensional analysis one can see that only the term proportional to C^2_{21} contributes to the singularity at the origin and we get (see the next section)

$$\zeta(s|H) = \frac{C^2_{21}}{4s} + \cdots \,, \qquad C^2_{21} = -\frac{2}{3} \,. \tag{3.6}$$

We conclude this section by studying the asymptotics of the spectral density associated with the operator H. We can define the spectral density via the spectral representation of the heat trace, namely

$$\mathrm{Tr}\, e^{-tH} = \int_0^\infty e^{-t\lambda}\, dN(\lambda) = \int_0^\infty d\lambda e^{-t\lambda}\rho(\lambda) \,. \tag{3.7}$$

For the polynomial interaction, the Tauberian theorems (see the Appendix) and the short-t leading terms of the heat-trace expansion give

$$N(\lambda) \simeq \lambda^{\frac{dQ+2d}{2Q}} \,, \qquad \rho(\lambda) \simeq \lambda^{\frac{dQ+2d}{2Q}-1} \,, \qquad \lambda \to \infty \,, \tag{3.8}$$

while for the exponential interaction, with d/Q an integer,

$$N(\lambda) \simeq \lambda^{\frac{d}{Q}} (\ln\lambda)^{d/Q} \,, \qquad \rho(\lambda) \simeq \lambda^{\frac{d}{Q}-1} (\ln\lambda)^{\frac{d}{Q}} \,, \qquad \lambda \to \infty \,. \tag{3.9}$$

In particular, when $Q = d$, one has

$$N(\lambda) \simeq \lambda\, (\ln\lambda) \,, \qquad \rho(\lambda) \simeq \ln\lambda \,, \qquad \lambda \to \infty \,. \tag{3.10}$$

In this last case the distribution of the eigenvalues of the operator H resembles the asymptotic behaviour which one meets in number theory, namely the asymptotic distribution of the non-trivial zeroes of the Riemann zeta function [22]. With regard to this important issue we refer the reader to the literature, mentioning the relevance of the method based on Cramer's V-function [36] and references therein. Other related papers are [37]-[38].

4 A simple model of confinement

In this section we investigate an explicit model, namely a massive scalar field defined on a flat spacetime $R \times R^3$ in an external static field described by a confining potential which is asymptotically exponential in two dimensions. In the Euclidean version, we may compactify the "time" coordinate and the *zeta* spatial coordinate, assuming periodic boundary conditions with periods β and l, respectively. As a result, the relevant operator reads

$$L = -\frac{d^2}{d^2\tau} - \frac{d^2}{dz^2} + H_2 + M^2 \,, \qquad H_2 = -\Delta_{\,2} + V(r) \,, \qquad V(r) = g^2 e^{\alpha^2 r^2} \,, \tag{4.1}$$

g and α being dimensional parameters. Making use of Poisson's re-summation formula, the heat trace can be written as

$$\mathrm{Tr}\, e^{-tL} = \frac{S\, e^{-tM^2}}{4\pi t}\ \mathrm{Tr}\, e^{-tH_2} + \dots , \tag{4.2}$$

where $S = \beta\, l$ and the dots stand for exponentially small terms in the parameter t

In this model, the zeta function can be computed by using the method described in the previous section, but one now obtains an expression which is different from (3.3), since the potential is defined only on R^2 and one needs to take the factor $e^{-tM^2/t}$ into account. As a result, we get

$$\begin{aligned} \zeta(s|L) &\sim \frac{S}{(4\pi)^2\Gamma(s)} \sum_n \int_0^\infty dt\, t^{s+n-3} \int_{R^2} dx\, \tilde{b}_n(x) e^{-tV(r)} \\ &= \sum_n \frac{\Gamma(s+n-2)}{(4\pi)^2\Gamma(s)} \int_{R^2} dx\, \tilde{b}_n(x) [V(r)]^{-(s+n-2)} , \end{aligned} \tag{4.3}$$

where the $\tilde{b}_n$ are related to the b_n in (3.4) by

$$\tilde{b}_n = \sum_{j+k=n} \frac{(-1)^k b_j M^{2k}}{k!} , \qquad n \geq 2 , \qquad \tilde{b}_0 = 1 , \qquad \tilde{b}_1 = -M^2 . \tag{4.4}$$

The $\tilde{b}_n$ have again the same structure as in Eq. (3.4), but now q can vanish, namely

$$\tilde{b}_n = \sum_{pq} \tilde{C}^n_{pq} r^p a^q e^{qbr^2} , \qquad 0 \leq p \leq 2(n-1) , \qquad 0 \leq q < n , \qquad n \geq 0 , \tag{4.5}$$

$$\tilde{C}^n_{00} = \frac{(-1)^n M^{2n}}{n!} . \tag{4.6}$$

The b_n coefficients which appear in Eq. (4.5) can be evaluated by making use of (1.9-1.10); the first non-trivial ones read

$$\begin{aligned} b_2 &= -\frac{2g\alpha e^{\alpha r^2}}{3}(1+\alpha r^2) , \\ b_3 &= -\frac{4g\alpha^2 e^{\alpha r^2}}{15}(2+4\alpha r^2+\alpha^2 r^4) + \frac{g^2\alpha^2 e^{2\alpha r^2}}{3} , \\ b_4 &= -\frac{8g\alpha^3 e^{\alpha r^2}}{105}(6+18\alpha r^2+9\alpha^2 r^4+\alpha^3 r^6) + \frac{2g^2\alpha^2 e^{2\alpha r^2}}{45}(7+38\alpha r^2+21\alpha^2 r^4) , \end{aligned} \tag{4.7}$$

from which we can read off the C^n_{pq} coefficients up to $n = 4$.

By integrating (4.3), the non-holomorphic contribution to the zeta function reads

$$\zeta(s|L) = \frac{S}{16\pi\Gamma(s)} \sum_{n\geq 0;pq} \tilde{C}^n_{pq} \frac{\Gamma(s+n-2)\Gamma(1+p/2)\, a^{-(s+n-q-2)}}{b^{1+p/2}\,(s+n-q-2)^{1+p/2}} . \tag{4.8}$$

Since in our specific example p is even, the zeta function has only poles of oder $p/2$. In particular, in a neighbourhood of $s = 0$ the pole structure is

$$\zeta(s|L) = \frac{S}{16\pi\alpha}\left[\frac{M^4}{2s} + \sum_{n=3}^{6} \frac{\tilde{C}^n_{2,n-2}\Gamma(n-2)}{\alpha s} + 2\sum_{n=3}^{6} \frac{\tilde{C}^n_{4,n-2}\Gamma(n-2)}{\alpha^2 s^2}\right] + \cdots \tag{4.9}$$

Thus, the zeta function $\zeta(s|L)$ is *not* regular at the origin: a pole of second order appears. Within a physical context (restricted most of the time to the realm of pseudo-differential operators in compact domains), this is a very unusual behaviour for the zeta function [19]-[21]. In these cases, as far as the one-loop effective action is concerned, the otherwise well established zeta function regularisation procedure needs to be modified [5, 39].

Our proposal, which extends in a natural way the one formulated in [19]-[21], consists in the introduction of an additional spectral function which depends on the order of the pole at the origin of the initial zeta function. Thus, in the case of a pole of order N, the auxiliary spectral function reads

$$\omega(s) = s^N \zeta(s|L)\,, \tag{4.10}$$

and the definition of the regularised determinant is generalised as

$$\ln \det \frac{L}{\mu^2} = -\frac{1}{(N+1)!} \lim_{s\to 0} \frac{d^{N+1}}{ds^{N+1}} \left[\mu^{2s}\omega(s)\right]\,, \tag{4.11}$$

with the normalisation chosen in such a way that when $\zeta(s|L)$ is regular at the origin, one does recover the ordinary definition of regularised functional determinant. This is an essential condition in order to preserve the well established properties defining the zeta function regularisation procedure.

Recalling our example before, we have seen that in this model a second-order pole will generically appear. Then, the new spectral function, which is regular at the origin, will be given by

$$\omega(s) = s^2 \zeta(s|L)\,. \tag{4.12}$$

We correspondingly define

$$\ln \det \frac{L}{\mu^2} = -\frac{1}{3!} \lim_{s\to 0} \frac{d^3}{ds^3} \left[\mu^{2s}\omega(s)\right]\,. \tag{4.13}$$

It has to be mentioned here that, within the context of a general theory of regularised products (see e.g. [27]), in the case when the related zeta function is not holomorphic at the origin but has a first-order pole —and we have stressed this to happen when logarithmic terms are present in the heat-trace asymptotics— a new definition of regularised product was proposed recently [40]. It reads

$$\prod_{k=1}^{\infty} \lambda_k \equiv \exp\left[-\,\mathrm{Res}\big(\frac{\zeta(s)}{s^2}\big)_{s=0}\right]\,, \qquad \zeta(s) = \sum_{k=1}^{\infty} \lambda_k^{-s}\,. \tag{4.14}$$

Recalling the definition of residue, it is straightforward to conclude that this prescription is equivalent to ours, Eq. (4.11). This is a further consistency check and inscribes our result in a very general context.

We conclude with the following remark. The one-loop renormalisation group equations associated with the presence of the renormalisation scale μ can be treated along the same lines as in Ref. [39]. This shows both the power and flexibility of the zeta-function method to easily cope with non-standard and unexpected situations, without ever losing contact with the fundamental issue of its applicability to actual physical problems. This means, in particular, that the results obtained with the method must be checked to be physically meaningful and to reproduce measured experimental values.

5 Conclusion

In this paper we have considered several examples of the determination of heat-kernel traces associated with operators of Laplace type defined on non-compact domains. For the sake of simplicity, we have restricted our analysis to R^d and to analytic but confining potentials, thus dealing with discrete spectra only. However, the adequacy of the procedure to treat more general settings has been exhibited. In particular, although for the sake of simplicity we have postponed the treatment of the case when d/Q is non-integer, with some extra effort this can be dealt with along the same lines. New branching points appear there.

We have shown that for confining potentials of exponential behaviour at infinity, the asymptotics of the heat-kernel trace contain generically logarithmic terms. As a consequence, the meromorphic structure of the associated zeta function develops higher-order poles as well as branching points. In particular, we have exhibited some cases where the zeta function is not regular at the origin.

In these situations, one is confronted with the non-trivial task of having to define the corresponding regularised functional determinant or the one-loop effective potential. In fact, in the example of an apparently reasonable model of confinement, constructed by means of an asymptotically exponential potential, we have proven that the meromorphic continuation of the zeta function already develops a higher-order pole at the origin. This, as far as we know, is an absolutely novel finding in the field, even more since it comes from such an apparently harmless model.

In order to deal with these special cases, we have proposed a generalisation of the zeta function regularisation procedure, consisting in the introduction of a new, auxiliary zeta function which is still regular at the origin, together with a correspondingly new definition for the zeta-regularised determinant (thus extending Ray and Singer's definition). This general prescription —which naturally extends particular cases already considered by some of us before [19]-[21]— turns out to be equivalent to the one recently proposed by Hirano et al. [40] in a more generic context of a theory of regularised products. In accordance with our fundamental aim never to abandon the already established connections with the physical world (e.g., the many uses of zeta regularisation in experimental physics), all the new quantities have been defined in such a way as to recover the celebrated results of zeta-function regularisation in the absence of poles at the origin.

Still pending is the task to construct an explicit general theory to deal with the whole class of Laplacian like operators in non-compact domains, what we leave for further work.

Acknowledgements

We thank the referee for valuable remarks and criticisms which have resulted in a significant improvement of this paper. This paper is an outcome of the collaboration program INFN (Italy)–DGICYT (Spain). EE has been also supported by DGICYT (Spain), project BFM2003-00620, and AGAUR (Generalitat de Catalunya), contract 2005SGR-00790.

A Appendix: Some useful formulae

Here we list some expressions that have been employed in the text. We start with the incomplete gamma function, useful in order to reveal the presence of logarithmic terms in the heat-kernel trace expansion. Its definition reads

$$\Gamma(s,t) = \int_t^\infty dy\, y^{s-1} e^{-y} = \Gamma(s) - \frac{t^s}{s} - t^s \sum_{r=1}^{\infty} \frac{t^n}{n!(s+n)}, \tag{A.1}$$

and thus

$$\Gamma(0,t) = -\ln t - \gamma - t - \frac{t^2}{4} + O(t^3)\,, \tag{A.2}$$

γ being Euler's constant. Taking the derivative of order n of $\Gamma(s,t)$ with respect to s, one gets

$$\frac{d^n}{ds^n}\Gamma(s,t) = \int_t^\infty dy\ (\ln y)^n y^{s-1} e^{-y}\,, \tag{A.3}$$

from which it follows that

$$\begin{aligned} \int_1^\infty dy\, (\ln y)^n y^{s-1} e^{-ty} &= \frac{d^n}{ds^n}\frac{\Gamma(s,t)}{t^s} \\ &= \sum_{k=0}^{n} f_k(s)\Gamma(s)\,\frac{(\ln t)^k}{t^s} - \frac{(-1)^n n!}{s^{n+1}} + O(t), \end{aligned} \tag{A.4}$$

where the $f_k(s)$ are computable functions. In particular,

$$f_n(s) = (-1)^n\,, \qquad f_{n-1}(s) = (-1)^{n+1}\psi(s)\,, \tag{A.5}$$

$\psi(s)$ being the digamma function. In the limit $s \to 0$ we finally have

$$\int_1^\infty dy\, (\ln y)^n y^{-1} e^{-ty} = \sum_{k=0}^{n+1} c_k\, (\ln t)^k + O(t)\,, \tag{A.6}$$

where

$$c_{n+1} = \frac{(-1)^{n+1}}{n+1}\,, \qquad c_n = (-1)^{n+1}\gamma\,. \tag{A.7}$$

With a view to the analytic continuation of zeta functions, the following formulas are useful too. If α is a complex number with positive real part, and β such that $\mathrm{Re}\ \beta > -1$, one has

$$\int_0^1 dt\, t^{\alpha-1} (\ln t)^\beta = \frac{(-1)^\beta \Gamma(\beta+1)}{\alpha^{\beta+1}}\,. \tag{A.8}$$

To prove this, it is sufficient to perform the change of variable $u = -\ln t$ and recall the definition of the Euler gamma function.

Furthermore, for the asymptotics of the spectral density for large λ, one has

$$t^{-s} = \int_0^\infty e^{-t\lambda} \frac{\lambda^{s-1}}{\Gamma(s)}\,. \tag{A.9}$$

Taking the derivative with respect to s,

$$\ln t\, t^{-s} = \int_0^\infty e^{-t\lambda}\lambda^{s-1}\left[-\frac{\ln\lambda}{\Gamma(s)} + \frac{\Gamma'(s)}{\Gamma(s)}\right], \tag{A.10}$$

and

$$\ln^2 t\, t^{-s} = \int_0^\infty e^{-t\lambda}\lambda^{s-1}\left[\frac{\ln^2\lambda}{\Gamma(s)} + 2\frac{\Gamma'(s)}{\Gamma^2(s)}\ln\lambda - \frac{\Gamma''(s)}{\Gamma(s)} - \frac{\Gamma'(s)}{\Gamma^2(s)}\right]. \tag{A.11}$$

The above identities are compatible with the Karamata Tauberian theorems, which can be stated as follows. Suppose we deal with

$$\int_0^\infty e^{-t\lambda} dN(\lambda) = K(t)\,. \tag{A.12}$$

i) If

$$K(t) \simeq A\, t^{-r}\,, \qquad t \to 0\,, \tag{A.13}$$

then

$$N(\lambda) \simeq A\frac{\lambda^r}{\Gamma(r+1)}\,, \qquad \lambda \to \infty\,. \tag{A.14}$$

ii) If

$$K(t) \simeq A\, t^{-r}\ln^N t\,, \qquad t \to 0\,, \tag{A.15}$$

then

$$N(\lambda) \simeq A\frac{\lambda^r \ln^N\lambda}{\Gamma(r+1)}\,, \qquad \lambda \to \infty\,. \tag{A.16}$$

References

[1] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta Regularisation Techniques with Applications*, World Scientific, Singapore (1994).

[2] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, Springer Verlag, Berlin (1995).

[3] A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Rept. **266**, 1 (1996).

[4] M. Bordag, E. Elizalde and K. Kirsten, J. Math. Phys. **37**, 895 (1996); M. Bordag, E. Elizalde, K. Kirsten and S. Leseduarte, Phys. Rev. **D56**, 4896 (1997); E. Elizalde, M. Bordag and K. Kirsten, J. Phys. **A31**, 1743 (1998); M. Bordag, E. Elizalde, B. Geyer and K. Kirsten, Commun. Math. Phys. **179**, 215 (1996).

[5] A.A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti and S. Zerbini, *Analytic Aspects of Quantum Fields*, World Scientific, Singapore (2003).

[6] D.B. Ray and I.M. Singer, Ann. Math. **98**, 154 (1973).

[7] S.W. Hawking, Commun. Math. Phys. **55**, 133 (1977).

[8] J.S. Dowker and R. Critchley, Phys. Rev. **D13**, 3224 (1976).

[9] B.S. DeWitt, *The Dynamical Theory of Groups and Fields*, Gordon and Breach, New York (1965).

[10] R.T. Seeley, Am. Math. Soc. Prog. Pure Math. **10**, 172 (1967).

[11] K. Kirsten, *Spectral Functions in Mathematics and Physics*, Chapman and Hall, CRC, London (2000).

[12] D.V. Vassilevich, Phys. Rept. **388**, 279 (2003).

[13] L. Parker and D.J. Toms, Phys. Rev. **D31**, 953 (1985).

[14] J. Jack and L. Parker, Phys. Rev. **D31**, 2439 (1985).

[15] M. Bordag, K. Kirsten and J.S. Dowker, Commun. Math. Phys. **182**, 371 (1996).

[16] G. Cognola and S. Zerbini, Lett. Math. Phys. **42**, 95 (1997).

[17] A.A. Bytsenko, G. Cognola and S. Zerbini, J. Phys. **A30**, 3543 (1997); A.A. Bytsenko, G. Cognola and S. Zerbini, Class. Quantum Grav. **14**, 615 (1997).

[18] G. Grubb, Nucl. Phys. Proc. Suppl. **B104**, 71 (2002).

[19] A.A. Bytsenko, E. Elizalde and S. Zerbini, Phys. Rev. **D64**, 105024 (2001).

[20] E. Elizalde, J. Phys. **A34**, 3025 (2001).

[21] A.A. Bytsenko, A.E. Gonçalves and S. Zerbini, Mod. Phys. Lett. **A16**, 1479 (2001).

[22] C. Nash, Ad. Applied Maths. **6**, 436 (1985).

[23] A. Voros, J. Phys. **A33**, 7423 (2000).

[24] J.S. Dowker and K. Kirsten, Analysis Appl. **3**, 45 (2005).

[25] C. Callias, Commun. Math. Phys. **88**, 357 (1983); C. Callias, Commun. Partial Diff. Eqs. **13**, 1113 (1988).

[26] K. Kirsten, P. Loya and J. Park, J. Math. Phys. **47** 043506 (2006).

[27] G. Illies, Commun. Math. Phys. **220**, 69 (2001).

[28] J. Jorgenson, S. Lang and D. Goldfeld, *Explicit formulas*, LNM **1593**, Berlin, Springer (1994); J. Jorgenson and S. Lang, *Basic Analysis of regularized series and products*, LNM **1564**, Berlin, Springer (1994).

[29] Yu. Manin, Asterisque **228**, 121 (1995).

[30] N. Dencker, Ark. Math. **24**, 59 (1986).

[31] B. Simon, Ann. Phys. (NY) **146**, 209 (1983).

[32] W. Müller, Commun. Math. Phys. **192**, 309 (1998).

[33] S. Solodhukin, Nucl. Phys. **B541**, 401 (1999).

[34] M. Bordag and D.V. Vassilevich, J. Phys. **A32**, 8247 (1999); M. Bordag, I.G. Pirozhenko and V.V. Nesterenko, J. Phys. **A38**, 11027 (2005).

[35] N.R. Khusnutdinov, Phys. Rev. **D73** 025003 (2006).

[36] H. Cramer, Math. Z. **4**, 104 (1919); J. Jorgenson and S. Lang, Math. Ann. **297**, 383 (1993).

[37] H.C. Rosu, Mod. Phys. Lett. **A18**, 1205 (2003).

[38] G. Sierra, J .Stat. Mech. **0512**, P12006 (2005).

[39] G. Cognola and S. Zerbini, Phys. Rev. **D69**, 024004 (2004).

[40] M. Hirano, N. Kurokawa, M. Wakayama, J. Ramanujan Math. Soc. **18**, 195 (2003).